

Conditional Symmetries, the True Degree of Freedom and G.C.T. Invariant Wave functions for the general Bianchi Type II Vacuum Cosmology

T. Christodoulakis[†], G.O. Papadopoulos[‡]

*University of Athens, Physics Department Nuclear & Particle Physics Section
GR-15771 Athens, Greece*

Abstract

The quantization of the most general Bianchi Type II geometry –with all six scale factors, as well as the lapse function and the shift vector, present– is considered. In an earlier work, a first reduction of the initial 6-dimensional configuration space, to a 4-dimensional one, has been achieved by the usage of the information furnished by the quantum form of the linear constraints. Further reduction of the space in which the wave function –obeying the Wheeler-DeWitt equation– lives, is accomplished by unrevealing the extra symmetries of the Hamiltonian. These symmetries appear in the form of –linear in momenta– first integrals of motion. Most of these symmetries, correspond to G.C.T.s through the action of the automorphism group. Thus, a G.C.T. invariant wave function is found, which depends on the only true degree of freedom, i.e. the unique curvature invariant, characterizing the hypersurfaces $t = \text{const.}$

e-mail: [†] tchris@cc.uoa.gr, [‡] gpapado@cc.uoa.gr

1

As it is well known, the quantum cosmology approximation, consists in freezing out all but a finite number of degrees of freedom of the gravitational field, and quantize the rest. This is done by imposing spatial homogeneity. Thus our –in principle– dynamical degrees of freedom, are the scale factors $\gamma_{\alpha\beta}(t)$, the lapse function $N(t)$ and the shift vector $N^a(t)$, of some Bianchi Type geometry.

The general Bianchi Type II cosmology, had been treated in [1], where a wave function –in terms of 4 combinations of the 6 $\gamma_{\alpha\beta}(t)$ – had been presented. According to the hints given in the discussion of [1], this reduction ought not to be the final one, since there were gauge degrees of freedom left.

In this sort communication, we present the desired final reduction and the only true degree of freedom is revealed as an argument of the wave function.

2

In [1], we had considered the quantization of an action corresponding to the most general Bianchi Type II cosmology, i.e. an action giving Einstein's field equations, derived from the line element:

$$ds^2 = (N^2(t) - N_a(t)N^a(t))dt^2 + 2N_a(t)\sigma_i^a(x)dx^i dt + \gamma_{\alpha\beta}(t)\sigma_i^\alpha(x)\sigma_j^\beta(x)dx^i dx^j \quad (1)$$

with:

$$\begin{aligned} \sigma^a(x) &= \sigma_i^\alpha(x)dx^i \\ \sigma^1(x) &= dx^2 - x^1 dx^3 \\ \sigma^2(x) &= dx^3 \\ \sigma^3(x) &= dx^1 \\ d\sigma^a(x) &= \frac{1}{2}C_{\beta\gamma}^a \sigma^\beta \wedge \sigma^\gamma \\ C_{23}^1 &= -C_{32}^1 = 1 \end{aligned} \quad (2)$$

see [2].

As is well known [3], the Hamiltonian is $H = N(t)H_0 + N^a(t)H_a$ where:

$$H_0 = \frac{1}{2}L_{\alpha\beta\mu\nu}\pi^{\alpha\beta}\pi^{\mu\nu} + \gamma R \quad (3)$$

is the quadratic constraint with:

$$\begin{aligned} L_{\alpha\beta\mu\nu} &= \gamma_{\alpha\mu}\gamma_{\beta\nu} + \gamma_{\alpha\nu}\gamma_{\beta\mu} - \gamma_{\alpha\beta}\gamma_{\mu\nu} \\ R &= C_{\lambda\mu}^{\beta} C_{\theta\tau}^{\alpha} \gamma_{\alpha\beta} \gamma^{\theta\lambda} \gamma^{\tau\mu} + 2C_{\beta\delta}^{\alpha} C_{\nu\alpha}^{\delta} \gamma^{\beta\nu} + 4C_{\mu\nu}^{\mu} C_{\beta\lambda}^{\beta} \gamma^{\nu\lambda} = C_{\mu\kappa}^{\alpha} C_{\nu\lambda}^{\beta} \gamma_{\alpha\beta} \gamma^{\mu\nu} \gamma^{\kappa\lambda} \end{aligned}$$

γ being the determinant of $\gamma_{\alpha\beta}$ (the last equality holding only for the Type II case), and:

$$H_a = C_{a\rho}^\mu \gamma_{\beta\mu} \pi^{\beta\rho} \quad (4)$$

are the linear constraints.

The quantities H_0 , H_a , are weakly vanishing [4], i.e. $H_0 \approx 0$, $H_a \approx 0$. For all class A Bianchi Types ($C_{\alpha\beta}^\alpha = 0$), it can be seen to obey the following first-class algebra:

$$\begin{aligned} \{H_0, H_0\} &= 0 \\ \{H_0, H_a\} &= 0 \\ \{H_a, H_\beta\} &= -\frac{1}{2} C_{\alpha\beta}^\gamma H_\gamma \end{aligned} \quad (5)$$

which ensures their preservation in time i.e. $\dot{H}_0 \approx 0$, $\dot{H}_a \approx 0$ and establishes the consistency of the action.

If we follow Dirac's general proposal [4] for quantizing this action, we have to turn H_0 , H_a , into operators annihilating the wave function Ψ .

In the Schrödinger representation:

$$\begin{aligned} \gamma_{\alpha\beta} &\rightarrow \hat{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} \\ \pi^{\alpha\beta} &\rightarrow \hat{\pi}^{\alpha\beta} = -i \frac{\partial}{\partial \gamma_{\alpha\beta}} \end{aligned} \quad (6)$$

satisfying the basic Canonical Commutation Relation (CCR) –corresponding to the classical ones:

$$[\hat{\gamma}_{\alpha\beta}, \hat{\pi}^{\mu\nu}] = -i \delta_{\alpha\beta}^{\mu\nu} = \frac{-i}{2} (\delta_\alpha^\mu \delta_\beta^\nu + \delta_\beta^\mu \delta_\alpha^\nu) \quad (7)$$

The quantum version of the 2 independent linear constraints has been used to reduce, via the method of characteristics [5], the dimension of the initial configuration space from 6 ($\gamma_{\alpha\beta}$) to 4 (combinations of $\gamma_{\alpha\beta}$), i.e. $\Psi = \Psi(q, \gamma, \gamma_{12}^2 - \gamma_{11}\gamma_{22}, \gamma_{12}\gamma_{13} - \gamma_{11}\gamma_{23})$, where $q = C_{\mu\kappa}^\alpha C_{\nu\lambda}^\beta \gamma_{\alpha\beta} \gamma^{\mu\nu} \gamma^{\kappa\lambda}$.

According to Kuchař's and Hajicek's [6] prescription, the “kinetic” part of H_0 is to be realized as the conformal Laplacian, corresponding to the reduced metric:

$$L_{\alpha\beta\mu\nu} \frac{\partial x^i}{\partial \gamma_{\alpha\beta}} \frac{\partial x^j}{\partial \gamma_{\mu\nu}} = g^{ij} \quad (8)$$

where x^i , $i = 1, 2, 3, 4$, are the arguments of Ψ . The solutions had been presented in [1]. Note that the first-class algebra satisfied by H_0 , H_a , ensures that indeed, all components of g^{ij} are functions of the x^i . The signature of the g^{ij} , is $(+, +, -, -)$ signaling the existence of gauge degrees of freedom among the x^i 's.

Indeed, one can prove [7] that the only gauge invariant quantity which, uniquely and irreducibly, characterizes a 3-dimensional geometry admitting the Type II symmetry group, is:

$$q = C_{\mu\kappa}^{\alpha} C_{\nu\lambda}^{\beta} \gamma_{\alpha\beta} \gamma^{\mu\nu} \gamma^{\kappa\lambda} \quad (9)$$

An outline of the proof, is as follows:

Let two exads $\gamma_{\alpha\beta}^{(1)}$ and $\gamma_{\alpha\beta}^{(2)}$ be given, such that their corresponding q 's, are equal. Then [7] there exists an automorphism matrix Λ (i.e. satisfying $C_{\mu\nu}^a \Lambda_a^{\kappa} = C_{\rho\sigma}^{\kappa} \Lambda_{\mu}^{\rho} \Lambda_{\nu}^{\sigma}$) connecting them, i.e. $\gamma_{\alpha\beta}^{(1)} = \Lambda_{\alpha}^{\mu} \gamma_{\mu\nu}^{(2)} \Lambda_{\beta}^{\nu}$. But as it had been shown in the appendix of [8], this kind of changes on $\gamma_{\alpha\beta}$, can be seen to be induced by spatial diffeomorphisms. Thus, 3-dimensional Type II geometry, is uniquely characterized by some value of q .

Although for full pure gravity, K  char [9] has shown that there are not other first-class functions, homogeneous and linear in $\pi^{\alpha\beta}$, except H_a , imposing the extra symmetries (Type II), allows for such quantities to exist –as it will be shown. We are therefore, naturally led to seek the generators of these extra symmetries –which are expected to chop off x^2 , x^3 , x^4 . Such quantities are, generally, called in the literature “Conditional Symmetries”.

The automorphism group for Type II, is described by the following 6 generators –in matrix notation and collective form:

$$\lambda_{(I)\beta}^a = \begin{pmatrix} \kappa + \mu & x & y \\ 0 & \kappa & \rho \\ 0 & \sigma & \mu \end{pmatrix} \quad (10)$$

with the property:

$$C_{\mu\nu}^a \lambda_a^{\kappa} = C_{\mu\sigma}^{\kappa} \lambda_{\nu}^{\sigma} + C_{\sigma\nu}^{\kappa} \lambda_{\mu}^{\sigma} \quad (11)$$

From these matrices, we can construct the linear –in momenta– quantities:

$$A_{(I)} = \lambda_{(I)\beta}^a \gamma_{\alpha\rho} \pi^{\rho\beta} \quad (12)$$

Two of these, are the H_a ,'s since $C_{(\rho)\beta}^a$ correspond to the inner automorphism subgroup –designated by the x and y parameters, in $\lambda_{(I)\beta}^a$. The rest of them, are the generators of the outer automorphisms and are described by the matrices:

$$\varepsilon_{(I)\beta}^a = \begin{pmatrix} \kappa + \mu & 0 & 0 \\ 0 & \kappa & \rho \\ 0 & \sigma & \mu \end{pmatrix} \quad (13)$$

The corresponding –linear in momenta– quantities, are:

$$E_{(I)} = \varepsilon_{(I)\beta}^a \gamma_{\alpha\rho} \pi^{\rho\beta} \quad (14)$$

The algebra of these –seen as functions on the phase space, spanned by $\gamma_{\alpha\beta}$ and $\pi^{\mu\nu}$ –, is:

$$\begin{aligned} \{E_I, E_J\} &= \tilde{C}_{IJ}^K E_K \\ \{E_I, H_a\} &= -\frac{1}{2} \lambda_a^\beta H_\beta \\ \{E_I, H_0\} &= -2(\kappa + \mu) \gamma R \end{aligned} \quad (15)$$

From the last of (15), we conclude that the subgroup of E_I ’s with the property $\kappa + \mu = 0$, i.e. the traceless generators, are first-class quantities; their time derivative vanishes. So let:

$$\tilde{E}_I = \{E_I : \kappa + \mu = 0\} \quad (16)$$

Then, the previous statement translates into the form:

$$\dot{\tilde{E}}_I = 0 \Rightarrow \tilde{E}_I = c_I \quad (17)$$

the c_I ’s being arbitrary constants.

Now, these are –in principle– integrals of motion. Since, as we have earlier seen, \tilde{E}_I ’s along with H_a ’s, generate automorphisms, it is natural to promote the integrals of motion (17), to symmetries –by setting the c_I ’s zero. The action of the quantum version of these \tilde{E}_I ’s on Ψ , is taken to be [6]:

$$\left. \begin{aligned} \widehat{\tilde{E}}_I \Psi &= \varepsilon_{(I)\beta}^a \gamma_{\alpha\rho} \frac{\partial \Psi}{\partial \gamma_{\beta\rho}} = 0 \\ \varepsilon_{(I)a}^a &= 0 \end{aligned} \right\} \Rightarrow \Psi = \Psi(q, \gamma) \quad (18)$$

The Wheeler-DeWitt equation now, reads:

$$5q^2 \frac{\partial^2 \Psi}{\partial q^2} - 3\gamma^2 \frac{\partial^2 \Psi}{\partial \gamma^2} + 2q\gamma \frac{\partial^2 \Psi}{\partial \gamma \partial q} + 5q \frac{\partial \Psi}{\partial q} - 3\gamma \frac{\partial \Psi}{\partial \gamma} - 2q\gamma \Psi = 0 \quad (19)$$

Note that:

$$\nabla_c^2 = \nabla^2 + \frac{(d-2)}{4(d-1)} R = \nabla^2$$

since we have a 2-dimensional, flat space, with contravariant metric:

$$g^{ij} = \begin{pmatrix} 5q^2 & q\gamma \\ q\gamma & -3\gamma^2 \end{pmatrix} \quad (20)$$

which is Lorentzian. This equation, can be easily solved by separation of variables; transforming to new coordinates $u = q\gamma^3$ and $v = q\gamma$, we get the 2 independent equations:

$$\begin{aligned} 16u^2 A''(u) + 4uA'(u) - cA(u) &= 0 \\ B''(v) + \frac{1}{2v}B'(v) - (\frac{1}{2v} + \frac{c}{4v^2})B(v) &= 0 \end{aligned} \quad (21)$$

where c , is the separation constant. Equation (19), is of hyperbolic type and the resulting wave function will still not be square integrable –under any measure. Besides that, the tracefull generators of the outer automorphisms, are left inactive –due to the non vanishing CCR with H_0 .

These two facts, lead us to deduce that there must still exist a gauge symmetry, corresponding to some –would be, linear in momenta– first-class quantity. Our starting point in the pursuit of this, is the third of (15). It is clear that we need another quantity –also linear in momenta– with an analogous property; the trace of $\pi^{\mu\nu}$, is such an object. We thus define the following quantity:

$$T = E_I - (\kappa + \mu)\gamma_{\alpha\beta}\pi^{\alpha\beta} \quad (22)$$

in the phase space –spanned by $\gamma_{\alpha\beta}$ and $\pi^{\mu\nu}$. It holds that:

$$\begin{aligned} \{T, H_0\} &= 0 \\ \{T, H_a\} &= 0 \\ \{T, E_I\} &= 0 \end{aligned} \quad (23)$$

because of:

$$\begin{aligned} \{E_I, \gamma\} &= -2(\kappa + \mu)\gamma \\ \{E_I, q\} &= 0 \\ \gamma_{\alpha\beta}\{\pi^{\alpha\beta}, q\} &= q \\ \gamma_{\alpha\beta}\{\pi^{\alpha\beta}, \gamma\} &= -3\gamma \end{aligned} \quad (24)$$

Again –as for \tilde{E}_I 's–, we see that since T , is first-class, we have that:

$$\dot{T} = 0 \Rightarrow T = \text{const} = c_T \quad (25)$$

another integral of motion. We therefore see, that T has all the necessary properties to be used in lieu of the tracefull generator, as a symmetry requirement on Ψ . In order to do that, we ought to set c_T zero –exactly as we did with the c_I 's, corresponding to \tilde{E}_I 's. The quantum version of T , is taken to be:

$$\hat{T} = \lambda_\beta^\alpha \gamma_{\alpha\rho} \frac{\partial}{\partial \gamma_{\beta\rho}} - (\kappa + \mu) \gamma_{\alpha\beta} \frac{\partial}{\partial \gamma_{\alpha\beta}} \quad (26)$$

Following, Dirac's theory, we require:

$$\hat{T}\Psi = \lambda_{\beta}^{\alpha}\gamma_{\alpha\rho}\frac{\partial\Psi}{\partial\gamma_{\beta\rho}} - (\kappa + \mu)\gamma_{\alpha\beta}\frac{\partial\Psi}{\partial\gamma_{\alpha\beta}} = (\kappa + \mu)(q\frac{\partial\Psi}{\partial q} - \gamma\frac{\partial\Psi}{\partial\gamma}) = 0 \quad (27)$$

Equation (27), implies that $\Psi(q, \gamma) = \Psi(q\gamma)$ and thus equation (19), finally, reduces to:

$$4w^2\Psi''(w) + 4w\Psi'(w) - 2w\Psi = 0 \quad (28)$$

where, for simplicity, $w \doteq q\gamma$. The solution to this equation, is:

$$\Psi = c_1 I_0(\sqrt{2q\gamma}) + c_2 K_0(\sqrt{2q\gamma}) \quad (29)$$

where I_0 is the modified Bessel function, of the first kind, and K_0 is the modified Bessel function, of the second kind, both with zero argument.

At first sight, it seems that although we have apparently exhausted the symmetries of the system, we have not yet been able to obtain a wave function on the space of the 3-geometries, since Ψ depends on $q\gamma$ and not on q only. On the other hand, the fact that we have achieved a reduction to one degree of freedom, must somehow imply that the wave function found must be a function of the geometry. This puzzle finds its resolution as follows. Consider the quantity:

$$\Omega = -2\gamma_{\rho\sigma}\pi^{\rho\sigma} + \frac{2C_{\mu\kappa}^{\alpha}C_{\nu\lambda}^{\beta}\gamma^{\kappa\lambda}\gamma^{\mu\nu}\gamma_{\alpha\rho}\gamma_{\beta\sigma} - 4C_{\mu\rho}^{\alpha}C_{\nu\sigma}^{\beta}\gamma_{\alpha\beta}\gamma^{\mu\nu}}{q}\pi^{\rho\sigma} \quad (30)$$

This can also be seen to be first-class, i.e.

$$\dot{\Omega} = 0 \Rightarrow \Omega = \text{const} = c_{\Omega} \quad (31)$$

Moreover, is a linear combination of T , \tilde{E}_I 's, and H_a 's, and thus $c_{\Omega} = 0$. Now it can be verified that Ω , is nothing but:

$$\frac{1}{N(t)}\left(\frac{\dot{\gamma}}{\gamma} + \frac{1}{3}\frac{\dot{q}}{q}\right) \quad (32)$$

So:

$$\gamma q^{1/3} = \vartheta = \text{constant} \quad (33)$$

Without any loss of generality, and since ϑ is not an essential constant of the classical system (see [10] and reference [18] therein), we set $\vartheta = 1$. Therefore:

$$\Psi = c_1 I_0(\sqrt{2}q^{1/3}) + c_2 K_0(\sqrt{2}q^{1/3}) \quad (34)$$

where I_0 is the modified Bessel function, of the first kind, and K_0 is the modified Bessel function, of the second kind, both with zero argument.

As for the measure, it is commonly accepted that, there is not a unique solution. A natural choice, is to adopt the measure that makes the operator in (28), hermitian –that is:

$$\mu(q) \propto q^{-1} \quad (35)$$

It is easy to find combinations of c_1 and c_2 so that the probability $\mu(q)|\Psi|^2$, be defined.

3

In this work, we were able to express the wave function for a Bianchi Type II Vacuum cosmology, in terms of the only true degree of freedom, i.e. the only curvature invariant (q) of the 3-geometry, under discussion. This was done by imposing the quantum versions of the first-class quantities \tilde{E}_I 's, T and Ω , as conditions on the wave function Ψ . A crucial point in this procedure, was the setting of the numerical values of \tilde{E}_I 's, T and Ω , equal to zero. The arguments in favor of this action, as far as \tilde{E}_I 's are concerned, are overwhelming since they are generators of automorphisms which, in turn, are induced by spatial G.C.T.s. Since T , is to replace the tracefull outer automorphism generator, it is mandatory to also set it zero. Once this has been done, Ω –being a linear function of H_a 's, \tilde{E}_I 's and T – is inevitably zero.

An additional argument for setting T and Ω zero, can be based on Dirac's theory: any first-class constraint is generator of a contact transformation. Moreover, any first-class quantity, is strongly equal to a linear combination of the constraints of the system. So, T and Ω , could be considered as generators of contact transformations i.e. as generators of covariance transformations, analogous to the automorphisms generated by (H_a, \tilde{E}_I) .

Note that putting the constant associated with Ω , equal to zero, amounts in restricting to a subset of the classical solutions, since c_Ω , is one of the two essential constants of Taub's solution. One could keep that constant, at the expense of arriving at a wave function with explicit time dependence, since then:

$$\gamma = q^{-1/3} \text{Exp} \left[\int c_\Omega N(t) dt \right]$$

We however, consider more appropriate to set that constant zero, thus arriving at a Ψ depending on q only, and decree its applicability to the entire space of the classical

solutions. Anyway this is not such a blunder, since Ψ is to give weight to all states, –being classical ones, or not.

So in conclusion, we see that not only the true degree of freedom is isolated but also the time problem has been solved –in the sense that a square integrable wave function Ψ is found. This is accomplished by unrevealing all the –hidden– gauge symmetries of the system. That wave function, is well defined on any spatially homogeneous 3-geometry.

A similar situation holds for Class A Types VI and VII; an object analogous to T also exists, and thus a reduction to the variables $\gamma q^1, \gamma q^2$ is possible with the help of this T and the 3 independent H_a 's ($q^1 = q, q^2 = C_{\beta\mu}^\alpha C_{\alpha\nu}^\beta \gamma^{\mu\nu}$, are the 2 independent curvature invariants, for these homogeneous 3-geometries). However, the passage to a G.C.T. invariant wave function i.e. $\Psi = \Psi(q^1, q^2)$, requires the knowledge of a first integral analogous to Ω , a thing that we lack –at present. The situation concerning Types VIII and IX, is more difficult: there are no outer automorphisms, and consequently no object analogous to T , exists. The H_a 's suffice to reduce the configuration space to q^1, q^2 and $q^3 = \text{Det}[m]/\sqrt{\gamma}$, now needed to specify the 3-geometry. The reduced supermetric is still Lorentzian, leading to a hyperbolic Wheeler-DeWitt equation. This fact is generally considered as a drawback, since it prohibits the ensuing wave function from being square integrable. Within the spirit of this work however, it can be taken as a positive sing; an object analogous to Ω may exist. In this case we would be able to reduce to a spacelike surface. We hope to return soon, with concrete results on these issues.

References

- [1] T. Christodoulakis, G. Kofinas, E. Korfiatis and A. Paschos,
Physics Letters B **390** (1997) 55-58.
- [2] M.P. Ryan & L.C. Shepley, “Homogeneous Relativistic Cosmologies”,
Princeton University Press, Princeton, New Jersey 1975.
- [3] G.E. Sneddon, Journal of Physics A **9** (1976) 2
M.A.H. MacCallum and A.H.T. Taub, Commun. Math. Phys. **25** (1972) 173
T. Christodoulakis and E. Korfiatis, Nuovo Cimento **109** B (1994) 11.
- [4] P.A.M. Dirac, Canadian Journal of Mathematics **2129** (1950);
Proceedings of Royal Society London A **246** (1958) 333;
Physical Review **114** (1959) 924;
“Lectures on Quantum Mechanics”, Yeshiva University, N.Y. 1965.
- [5] P.R. Carabedian “Partial Defferential Equations”, Chelsea, New York, 1986.
- [6] K.V. Kuchar and P. Hajiceck, Physical Review D **41** (1990) 1091;
Journal of Mathematical Physics **31** (1990) 1723.
- [7] T. Christodoulakis and E. Korfiatis,
“Kuchar’s Physical Variables and Automorphism Inducing Diffeomorphisms in Class
A Spatially Homogeneous Geometries”, University of Athens, Preprint 6, 1994.
- [8] T. Christodoulakis, G. Kofinas, E. Korfiatis and A. Paschos,
Physics Letters B **419** (1998) 30-36.
- [9] K.V. Kūchar, Journal of Mathematical Physics **23**(9) (1982) 1647-1661.
- [10] T. Christodoulakis, G. Kofinas, E. Korfiatis, G.O. Papadopoulos and A. Paschos,
gr-qc/0008050.